INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 4, 1027–1041 (1984)

ON THE CRITICAL PROBLEM OF F. D. PRESSURE TREATMENT FOR LAMINAR FLOWS CONFINED BY PERMEABLE WALLS

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SUMMARY

In the present work the viscous (low Reynolds) flow in plane ducts confined by permeable walls has been studied. A simple model of the filtrating walls has been used, with the normal velocity component proportional to the pressure jump across the wall, resulting in a non-standard boundary value Navier–Stokes problem.

A critical analysis of the appropriate boundary condition and pressure problem has led to the conclusions of employing a simple explicit finite volume approach, and of avoiding the use of higher order finite difference schemes. In this paper a special emphasis on the structure of the involved computational matrices has been given to illustrate the chosen algorithm. The latter yields a steady state solution that is second order accurate in space, and it has an accuracy in time of order $\leqslant \Delta t$ (the time step), due to the explicit treatment of the velocity boundary conditions along the membrane. The model has been tested to study the effects of the inlet/outlet conditions, Reynolds number and filtrating wall constant.

KEY WORDS F. D. for Navier–Stokes Mass Transfer Pressure Treatment

INTRODUCTION

Membrane technologies play a very important role in a variety of industrial, biomedical and biological applications. Confined flows by permeable walls are the object of detailed studies in environmental and chemical engineering. In most situations the performance of filters, the characteristics of sedimentation processes etc. are strongly influenced by the dynamics of the properties of such filtrating devices. The latter depend on heat and mass transfer as well as correlated electrochemical processes etc.

The knowledge of the fluid dynamics plays a fundamental role for a deep understanding of these wall-controlled phenomena. In most of the situations flow details are required in regions where viscous effects are not negligible, moreover the associated Reynolds numbers do not allow a simplification of the Navier–Stokes equations that have here been applied in their complete form.

In the present work the fundamental difficulties of treating boundary value problems (BVP) in the presence of a membrane have been examined for two-dimensional geometries. A special emphasis has been given to the problem of pressure determination and to develop a numerical algorithm that has successfully been tested on a personal computer, the HP-9826.

0271–2091/84/11 1027–15$01.50

Received 5 August 1983
Revised 11 November 1983

THE MODEL

Differential model

In the present work the laminar incompressible plane flow motion in the presence of a filtrating wall has been studied by applying the classical incompressible Navier–Stokes (NS) equations:

$$\left.\begin{aligned} \mathbf{v}_t &= \mathbf{r} - \nabla p \\ \nabla\cdot\mathbf{v} &= 0 \end{aligned}\right\} \text{ in } \Omega \tag{1}$$

where

$$\mathbf{r}(\mathbf{v}) = -\mathbf{v}\cdot\nabla\mathbf{v} + \nabla^2\mathbf{v}/Re$$

On $B\Omega$ (the contour of the domain Ω) different types of boundary conditions have to be satisfied by the solution. Let $B\Omega$ be equal to

$$B\Omega = \bigcup_{i=1,4} B\Omega_i$$

with

$$i \neq j \Rightarrow B\Omega_i \cap B\Omega_j = \varnothing$$

where $B\Omega_1$, $B\Omega_2$, $B\Omega_3$, $B\Omega_4$ represent respectively the inflow, outflow, membrane and solid wall boundaries (see Figure 1).

For a simple modelling of the membrane the normal velocity component is assumed to be proportional to the pressure jump across the membrane;[1] i.e.

$$\left.\begin{aligned} \mathbf{s}\cdot\mathbf{v} &= 0 \\ \mathbf{n}\cdot\mathbf{v} &= kp \end{aligned}\right\} \text{ on } B\Omega_3 \tag{2}$$

where $\mathbf{s}$ and $\mathbf{n}$ are, respectively, the unit tangent and normal vectors, and $k \geq 0$ is the filtrating constant.

Some physical and mathematical remarks on the pressure problem

To understand the motivation of the selected solution algorithm (that will be discussed in the next section), it is useful to make some physical and mathematical considerations on the problem of the pressure determination.

The model described above introduced a non-standard BVP. In most of the classical boundary

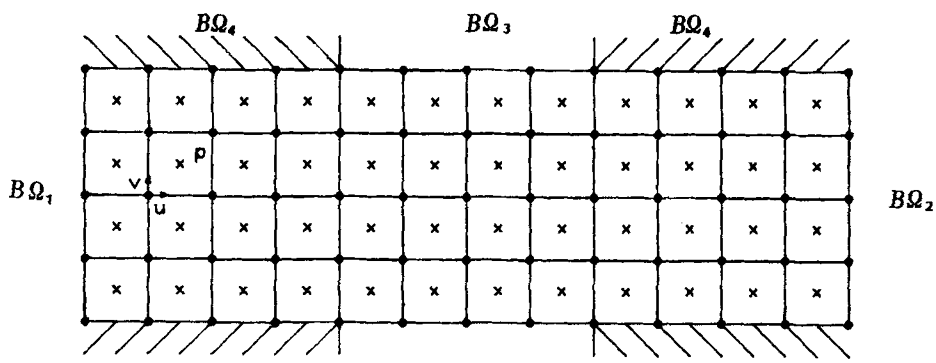

Figure 1. Computational geometry

value NS (BVNS) formulations, the pressure field is determined only by a constant. This is no longer valid in the present model, as equation (2) clearly indicates.

To the authors' knowledge theoretical closure aspects (such as existence, uniqueness, well-posedness), connected with the model problem (equations (1) and (2)), are not yet resolved. However such a remark still applies when strong solutions of other classical BVNS problems are sought.[2–5]

The theoretical closure of the present problem is not immediate even in a weak sense. The reason being that the velocity field is generally the only unknown of the classical weak formulations, and/or the pressure is introduced to satisfy the solenoidality constraint. Moreover the boundary conditions imposed to determine such a pressure are not the 'exact' ones.[3] In standard BVNS problems this may not be a critical point since (by using an appropriate algorithm) the computation of a pseudo-pressure does not affect the (asymptotic) convergence of the velocity vector field.

In conclusion, the peculiar pressure–velocity correlation along the membrane (equation (2)) would require further investigations in the theory of NS weak formulation, but this is outside the scope of the present work. On the other hand, for a correct numerical modelling, it is necessary to analyse some of the fundamental questions that are typical of weak/strong formulations. More specifically the meaning of what it is (or could be) intended for the 'exact' pressure b.c. is directly related to such questions. It is indeed useful to characterize once and for all the consequences of some smoothness assumptions when strong solutions of general BVNS problems are assumed to exist. This is also instructive for the increasing use of higher order finite difference schemes as well as self-adaptive algorithms that are mainly based on the existence and regularity of higher order derivatives of the original equations.[6]

It is common practice to consider the following Poisson-like equation for p (formally obtainable by taking the divergence of the momentum equation);

$$\nabla^2 p = \nabla \cdot \mathbf{r} \tag{3}$$

Paradoxes may arise if the b.c.s for the above equation are not properly prescribed.

If smoothness assumptions are extended to the (solid) boundary, from equation (1) one has:

$$\lim_{\mathrm{P} \to \mathrm{P_W}} (\mathbf{v}_t - \mathbf{r} + \nabla p) = (\mathbf{v}_t - \mathbf{r} + \nabla p)_{\mathrm{W}} = (\mathbf{v}_t)_{\mathrm{W}} - (\mathbf{r})_{\mathrm{W}} + (\nabla p)_{\mathrm{W}} = 0 \tag{4}$$

where $\mathrm{P} \in \Omega$ and $\mathrm{P_W} \in B\Omega$.

One might be tempted to use such an equation to obtain $(\nabla p)_{\mathrm{W}}$. Indeed deducing $\mathbf{r}$ as the limit from the interior, when $\mathbf{v}_t$ is assigned on $B\Omega$, equation (4) formally allow one to determine the pressure boundary conditions. However the validity of the equality established in equation (4) by the limit process is rather dubious and, on solid walls, there are no physical justifications for it, at least for the tangential component. A number of inconsistencies are produced as a consequence of the assumptions implied in equation (4) (often accepted or made by several authors[7,8]). Indeed the normal derivative $(\mathbf{n} \cdot \nabla p)_{\mathrm{W}}$ can be obtained from equation (4) and therefore the pressure is determined up to a constant. Incompatibility effects can arise because the tangential derivative of the computed p does not necessarily equal $(\mathbf{s} \cdot \nabla p)_{\mathrm{W}}$, as obtained from equation (4): the freedom in the assignment of $(\mathbf{s} \cdot \mathbf{v}_t)_{\mathrm{W}}$ allows one to select a value of the tangential velocity different from the one obtained by extrapolating (assuming continuity) the interior values.

By assuming further smoothness properties, the redundancy of the b.c. for ∇p (as obtained from equation (4)) could be formally eliminated in the way that will be shown next. But another paradox may arise. By taking the gradient of equation (3), the determination of the pressure forces $\mathbf{g}(\equiv \nabla p)$

can be reduced to the solution of the following uncoupled elliptic equations:

$$\nabla^2 \mathbf{g} = \nabla(\nabla \cdot \mathbf{r}), \quad \text{in } \Omega \tag{5}$$

with Dirichlet boundary condition on $B\Omega$ given by equation (4).

By definition $\mathbf{g}$ is a conservative vector field, i.e. $\nabla \times \mathbf{g} = 0$ in Ω. By taking the curl of equation (5) it results that $\nabla \times \mathbf{g}$ is harmonic. Therefore the solution of equation (5) yields a conservative field if and only if

$$\nabla \times \mathbf{g} = 0, \quad \text{on } B\Omega \tag{6}$$

(if Ω is a simply connected region).

It can be shown that by using only the normal component of equation (4), together with equation (6), boundary conditions (for $\mathbf{g}$) can be obtained that guarantee a lamellar $\mathbf{g}$ field as the solution of equation (5). However a paradox may arise because a value of $(\mathbf{s}\cdot\mathbf{g})_W$ different from $(\mathbf{s}\cdot\mathbf{r})_W-(\mathbf{s}\cdot\mathbf{v}_t)_W$, as obtained from equation (4), may result, for the freedom in the assignment of $(\mathbf{s}\cdot\mathbf{v}_t)_W$.

In conclusion the above paradoxes indicate that some smoothness properties have to be removed. Physical considerations (mass conservation) and the knowledge of the theory of partial differential equations suggest that continuity properties of the normal velocity components on $B\Omega$ must be maintained. Hence smoothness assumptions are relaxed for the tangential velocity component.

For the specific membrane confined flow only the boundary conditions for the normal velocity component are modified with respect to the classical BVNS problems. The above conclusions still apply as far as the tangential pressure gradient is concerned. The limit of the normal component of equation (4) along the membrane has been assumed in the form

$$(k(p_t)_-)_W-(\mathbf{n}\cdot\mathbf{r})_W + (p_n)_W = 0 \tag{7}$$

Equation (7) is still a Neumann condition for p having defined the first term as the limit of the left-time derivative, and again most of the previous considerations remain.

Finite volume equations and computational matrices structures

In the light of the above considerations, to avoid dubious pressure boundary conditions, the pressure has not been computed by solving a Poisson-like equation. Instead an algebraic pressure equation has been solved by imposing the condition that the mass is conserved in every computational volume (of course in the limit of $\Delta x, \Delta y \to 0$ it can be shown that the algebraic pressure equation is consistent with a Poisson-like one).

The governing equations have been discretized according to a local balance approach with a particular choice of finite volumes.[9] With such a choice no dubious boundary conditions are imposed and no special treatment is required along the membrane. The computational domain has been discretized using a grid with the following properties (see Figure 1): (i) constant step size; (ii) cell aspect ratio equal to one; (iii) both unknown velocity components defined at the grid nodes, and pressure staggered both along x and y; (iv) physical boundaries not staggered with respect to grid lines.

The finite difference equations have been obtained choosing as control volume (for both velocity components) a square cell centered around a grid node, and for mass conservation the natural grid cell centered around the 'pressure' nodes. The assumed finite difference formulae for internal nodes (i,j) are:

$$u_{ij}^{np} = u_{ij}^{n} - \alpha[(u_{ipj}^{n} + u_{ij}^{n})^2 - (u_{ij}^{n} + u_{imj}^{n})^2 + (u_{ijp}^{n} + u_{ij}^{n})(v_{ijp}^{n} + v_{ij}^{n}) - (u_{ij}^{n} + u_{ijm}^{n})(v_{ij}^{n} + v_{ijm}^{n})] \\ + \beta(u_{ipj}^{n} + u_{imj}^{n} - 4u_{ij}^{n} + u_{ijp}^{n} + u_{ijm}^{n}) - 2\alpha(p_{ij}^{np} + p_{ijm}^{np} - p_{imj}^{np} - p_{imjm}^{np}) \tag{8}$$

$$v_{ij}^{np} = v_{ij}^{n} - \alpha[(u_{ipj}^{n} + u_{ij}^{n})(v_{ipj}^{n} + v_{ij}^{n}) - (u_{ij}^{n} + u_{imj}^{n})(v_{ij}^{n} + v_{imj}^{n}) + (v_{ijp}^{n} + v_{ij}^{n})^2 - (v_{ij}^{n} + v_{ijm}^{n})^2] \\ + \beta(v_{ipj}^{n} + v_{imj}^{n} - 4v_{ij}^{n} + v_{ijp}^{n} + v_{ijm}^{n}) - 2\alpha(p_{ij}^{np} + p_{imj}^{np} - p_{ijm}^{np} - p_{imjm}^{np}) \tag{9}$$

$$u_{ipj}^{np} + u_{ipjp}^{np} - u_{ij}^{np} - u_{ijp}^{np} + v_{ijp}^{np} + v_{ipjp}^{np} - v_{ij}^{np} - v_{ipj}^{np} = 0 \tag{10}$$

where $ip = i + 1$, $im = i - 1$, and similarly for the js; $\alpha = \Delta t/4\Delta x$, $\beta = \Delta t/Re\Delta x^2$, $np = n + 1$. It can be shown that equations (8)–(10) are second order accurate in space.

The assumed finite difference boundary conditions are:

$$\left.\begin{aligned} \mathbf{n}\cdot(\mathbf{v}_{\mathrm{m}}^{np})_{ij} &= (k_i p_{ij}^{n} + k_{im} p_{imj}^{n})/2 \\ \mathbf{s}\cdot(\mathbf{v}_{\mathrm{m}}^{np})_{ij} &= 0 \end{aligned}\right\} \quad \text{on } B\Omega_3 \tag{11}$$

where $\mathbf{v}_{\mathrm{m}}$ is the velocity on the membrane.

On $B\Omega_4$ no slip b.c.s are imposed; on $B\Omega_1$ and $B\Omega_2$ the following conditions have been imposed:

$$\left.\begin{aligned} \mathbf{v} &= \mathbf{b}_1 = \text{given} \\ p &= p_1 = \text{given} \end{aligned}\right\} \quad \text{on } B\Omega_1$$

$$\mathbf{v} = \alpha_2 \mathbf{b}_2, \qquad \text{on } B\Omega_2$$

where $\mathbf{b}_2$ is an assumed shape function and α_2 is a real parameter so as to satisfy global mass conservation. A parabolic shape function can be reasonably assumed for a sufficient length of the impermeable wall ducts.

In a 'quasi'-matrix form the above equations are

$$\begin{pmatrix} \mathbf{I} & \Delta t\mathbf{G} & \mathbf{0} \\ \mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{v} \\ p \\ \mathbf{v}_{\mathrm{m}} \end{pmatrix}^{np} = \begin{pmatrix} \mathbf{I} + \Delta t\mathbf{A} & \mathbf{0} & \mathbf{A}_{\mathrm{m}} \\ \mathbf{0} & \mathbf{0} & -\mathbf{B}_{\mathrm{vm}} \\ \mathbf{0} & -\mathbf{M}_{\mathrm{p}} & \mathbf{0} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{v} \\ p \\ \mathbf{v}_{\mathrm{m}} \end{pmatrix}^{n} + \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix} \tag{12}$$

where $\mathbf{B}$ and $\mathbf{G}$ are linear operators (corresponding to the 'discretized' divergence and gradient operators); $\mathbf{A}(\mathbf{v})$ is a non-linear operator (the discretization of $\mathbf{r}$); $\mathbf{B}_{\mathrm{vm}}$ and $\mathbf{M}_{\mathrm{p}}$ represent the effects of the membrane; c_1 and c_2 account for non-conservative external forces and boundary conditions, respectively.

From equations (8)–(10) observe that $G_{ij} = -B_{ji}$ (this can also be verified by applying graph theory representation). Hence $\mathbf{G} = -\mathbf{B}^{\mathrm{T}}$ and therefore the finite difference forms of the divergence and $-$ gradient operators are still adjoint.

The pressure equation

From equation (12) observe that the velocity and pressure unknowns are the solution of the following system:

$$\begin{pmatrix} \mathbf{I} & -\Delta t\mathbf{B}^{\mathrm{T}} \\ \mathbf{B} & \mathbf{0} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{v} \\ p \end{pmatrix}^{np} = \begin{pmatrix} \mathbf{I} + \Delta t\mathbf{A} & \mathbf{0} & \mathbf{A}_{\mathrm{m}} \\ \mathbf{0} & \mathbf{0} & -\mathbf{B}_{\mathrm{vm}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{v} \\ p \\ \mathbf{v}_{\mathrm{m}} \end{pmatrix}^{n} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{13}$$

Let $\mathbf{T}$ be the following non-singular matrix:

$$\mathbf{T} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{B} & \mathbf{I} \end{pmatrix}$$

The matrix form of the pressure equation can be obtained by premultiplying equation (13) by $\mathbf{T}$, obtaining

$$\begin{pmatrix} \mathbf{I} & -\Delta t\mathbf{B}^{\mathrm{T}} \\ \mathbf{0} & \Delta t\mathbf{B}\mathbf{B}^{\mathrm{T}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{v} \\ p \end{pmatrix}^{np} = \begin{pmatrix} \mathbf{I}+\Delta t\mathbf{A} & \mathbf{0} \\ -\mathbf{B}(\mathbf{I}+\Delta t\mathbf{A}) & \mathbf{0} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{v} \\ p \end{pmatrix}^{n} + \begin{pmatrix} c'_1 \\ c'_2 \end{pmatrix} \tag{14}$$

Because of the reducibility of the above system the following pressure equation has to be solved:

$$\Delta t\mathbf{B}\cdot\mathbf{B}^{\mathrm{T}}\cdot\mathbf{p}^{np} \triangleq \mathbf{M}\mathbf{p}^{np} = \mathbf{B}(\mathbf{I}+\Delta t\mathbf{A})\mathbf{v}^{n} + c'_2 \tag{15}$$

The algebraic properties of the pressure matrix **M** and the correlation of the above system to the equivalent differential BVP play an important role in the proposed algorithm. The algebraic properties of **M** strictly affect the computational effort to obtain p. First of all symmetry and semipositive definiteness of **M** are guaranteed by definition. Secondly with an appropriate reordering (separation of odd/even pressure unknowns, see Figure 2) it becomes a two-block-diagonal matrix and it satisfies property A. Hence two uncoupled systems for odd and even ps are obtained with obvious gain in the required computational effort.

In matrix form one has:

$$\begin{aligned} \mathbf{M}'p'^{np} &= Q' \\ \mathbf{M}''p''^{np} &= Q'' \end{aligned} \tag{16}$$

where the odd/even pressure matrices ($\mathbf{M}'/\mathbf{M}''$) are still symmetric and semipositive definite.

Observe that such a decoupling can be exploited readily when symmetries are present. Indeed, instead of halving the domain Ω, one can equivalently solve for only one of the two pressure systems, and obtain the other one by symmetry, provided that an even number of cells is chosen in the direction normal to the centre line. This approach has the advantage that one can avoid any numerical treatment of the symmetry boundary conditions.

It can be shown that the existence of the solution, as well as the solenoidality of the velocity field, is guaranteed by imposing

$$\left.\begin{aligned} \sum_{ij}(m_{ij})' &= 0 \\ \sum_{ij}(m_{ij})'' &= 0 \end{aligned}\right\} \text{ on } B\Omega$$

where m_{ij} is the mass flux.

To rule out zero eigenvalues (vanishing of an eingenvalue is typical of physical situations where p is determined only by a constant) at least a reference pressure has to be imposed in an even and odd cell. This makes the above matrices positive definite.

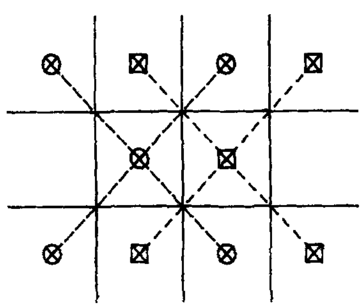

Figure 2. Even/odd pressure decoupling

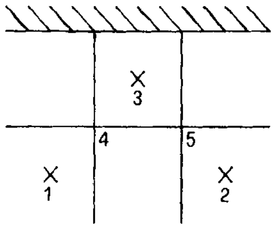

Figure 3. Wall cell for pressure boundary condition

The advantages of these algebraic properties can thus be exploited in the computational effort for solving the even and odd pressure system (equations (16)).

The difference formulae chosen for evaluating the pressure gradient in equations (8) and (9) guarantee that the two systems do not get out of phase. Indeed the 'discrete' pressure gradient is not affected by the reference values, since it is evaluated as the (vector) sum of the two gradients along the diagonals of the momentum control volume, each of which is obtained by using only even or odd pressure values. Thus the effects of an incorrect choice of the reference pressures are eliminated. Furthermore the 'discrete' pressure/velocity correlation along the filtrating wall (equation (11)) also eliminates the possibilities that the two pressure systems get out of phase, because the membrane velocity is calculated as the average of the two pressures at two adjacent p-nodes.

It is worth noticing that each of these systems is consistent with a coarser mesh discretization of an elliptic pressure equation, in agreement with the results of others[11] that showed the advantages of using a coarser discretization for evaluating pressure gradients to guarantee numerical kinetic energy conservation.

It is interesting to discuss the structure of the rows of $\mathbf{M}'(\mathbf{M}'')$ corresponding to the mass balance along the boundary cells to understand in differential terms the equivalent boundary conditions for p. Near a wall (see Figure 3) one has

$$-2p_3 + p_1 + p_2 = ((\mathbf{Av}^n)_5^x - (\mathbf{Av}^n)_4^x - (\mathbf{Av}^n)_5^y - (\mathbf{Av}^n)_4^y + (\mathbf{Bv}^n)_3/\Delta t)/\Delta Y \tag{17}$$

It is simple matter to verify that the above equation amounts to imposing the normal derivative boundary condition for p, given in equation (3). No tangential pressure derivative on the boundary is involved, consequently the danger of inconsistent results is avoided.

Observe that once the odd (even) pressures are obtained, half of the unknown pressure gradient components are numerically determined on the grid nodes. This is in the spirit of the solution of equation (5).

Moreover, with the present scheme the two discretized pressure derivatives are obtained by applying central difference formulae that operate separately on odd and even pressure unknowns. An important consequence is that if the odd/even pressures are determined only by a constant the velocity field is still correctly computed. This allows a simple optimization of the iterative solution by assuming the vanishing of an appropriate norm of the residual of the pressure equation as a constraint with the target of the minimum of $\| p^{np} - p^n \|$.[12,13]

Regularization method

A successive over-relaxation technique (SOR), with an optimized relaxation factor to improve the convergence rate,[14] has been employed for the pressure equation (15).

For the algebraic properties of $\mathbf{M}$ such a convergence is always guaranteed. However as the mesh size decreases the spectral radius of the iteration matrix approaches one and the rate of convergence becomes critical.

To reduce such problem, a regularization method has been employed, yielding the following modified equation:

$$(\mathbf{M} + \sigma \mathbf{I})\mathbf{p}^{np} = \mathbf{M}^*, \quad \mathbf{p}^{np} = \mathbf{q} + \sigma \mathbf{p}^n \tag{18}$$

$\mathbf{M}^*$ is still symmetric, positive definite, with a two-block-diagonal structure. Moreover it has a strong diagonal dominance with an obvious increase in the rate of convergence of the SOR algorithm.

Observe that by such an approach a pseudo-incompressible Navier–Stokes problem is solved (in analogy with Chorin's method), since the solenoidality is reached only at steady state. Indeed the mass conservation equation changes into the following balance equation:

$$\nabla \cdot \mathbf{v} = -\alpha p_t$$

where $\alpha = \sigma \Delta t$.

The optimum value of the regularization parameter σ is the one that minimizes the total number of iterations $N \times I$, where N and I are, respectively, the number of time steps to reach the steady state and the characteristic number of SOR iterations for the pressure.

The previous considerations show the strong correlations between matrix regularization, penalty, mass balance perturbation (Chorin) methods, etc.

RESULTS

The Newtonian flow motion confined by filtrating walls in two-dimensional planar geometries has been studied by applying an algorithm that exploits the simplicity of a fully explicit method. The effects of different filtrating constant and Reynolds number have been explored. The values of the parameters are given in Table I. The domain has been discretized with constant spacing in x and y, and all the results here discussed have been obtained on a grid consisting of 36×12 cells. In all cases a fully developed laminar (FDL) flow has been assumed at inflow and outflow: a FDL flow can be reasonably assumed in these regions, when $Re \leq 10$, because of the selected lengths of the inlet and outlet ducts. For all cases, but for $Re = 100$, the computed results show an excellent agreement of the velocity and pressure distributions with the FDL flow solution (note that the lack of orthogonality of the streamlines along the membrane is only due to poor graphics). All the computations were performed on a personal computer HP-9826, and the CPU time was of the order of one hour per case.

Table I. Computational parameters

Case No.	Re	$\bar{K}$
1	10	10^{-3}
2	10	$10^{-2.75}$
3	10	$10^{-2.5}$
4	10	$10^{-2.25}$
5	10	10^{-2}
6	1	10^{-3}
7	100	10^{-3}

Effects of the filtrating constant

Figures 4–8 show the computed pressure and streamline contour plots for different values of the filtrating constant k (ranging from 10^{-3} to 10^{-2}), by keeping constant both the inlet reference pressure and mass flow rate. Under these conditions it is observed that as k increases a flow reversal arises, as Figure 8 clearly shows. In Figure 8 the shear stress (τ) at the wall is plotted. The computed results show that for small values of k the variation of τ along x is negligible, but it becomes large as k increases, especially near the transition points (from solid to permeable walls and vice versa). In the central zone of the membrane a linear variation appears. Furthermore it is interesting to note a similarity of the profiles for the different values of k. Finally Figure 8 shows that a change in the sign of the shear stress can arise even if there is no flow reversal at the outlet.

Effect of the Reynolds number

In Figures 4, 10–12 the effects of the Reynolds number (values ranging from 1 to 100) are shown for a constant value of k, and fixed values of the reference pressure and mass flow rate at inlet. The results indicate that a change in the Reynolds number affects mainly the pressure field, with slight effects on the deformation rate along the wall (see Figure 12).

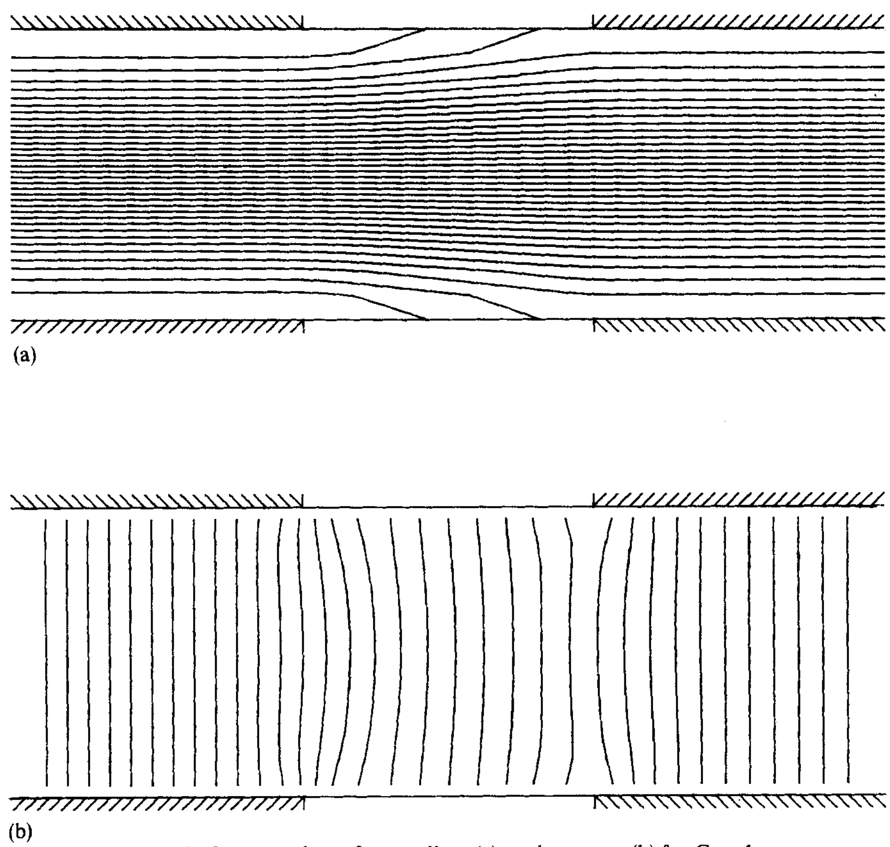

Figure 4. Contour plots of streamlines (a), and pressure (b) for Case 1

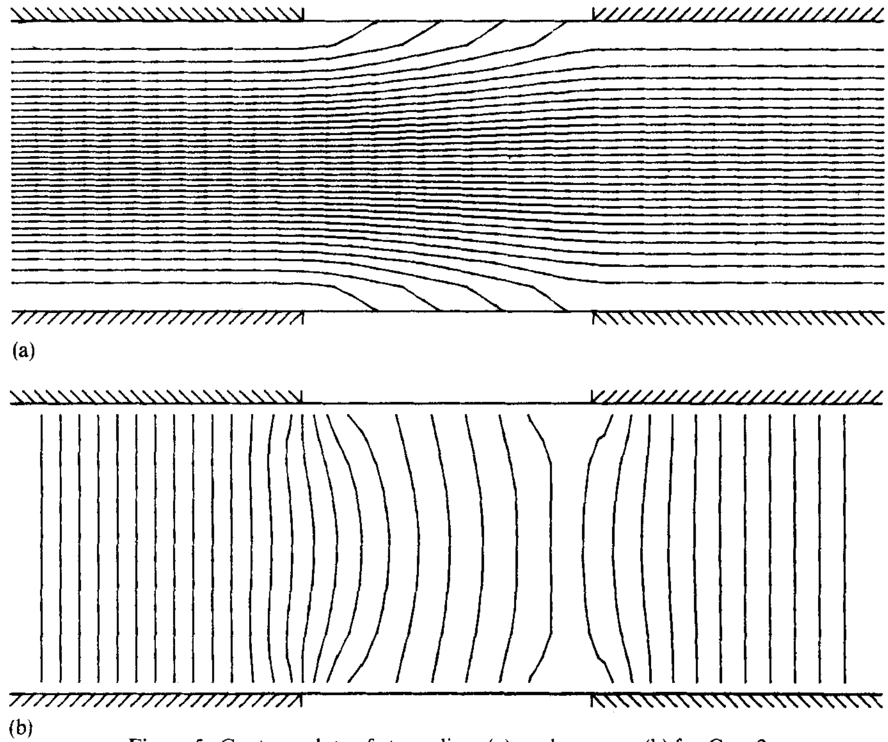

Figure 5. Contour plots of streamlines (a), and pressure (b) for Case 2

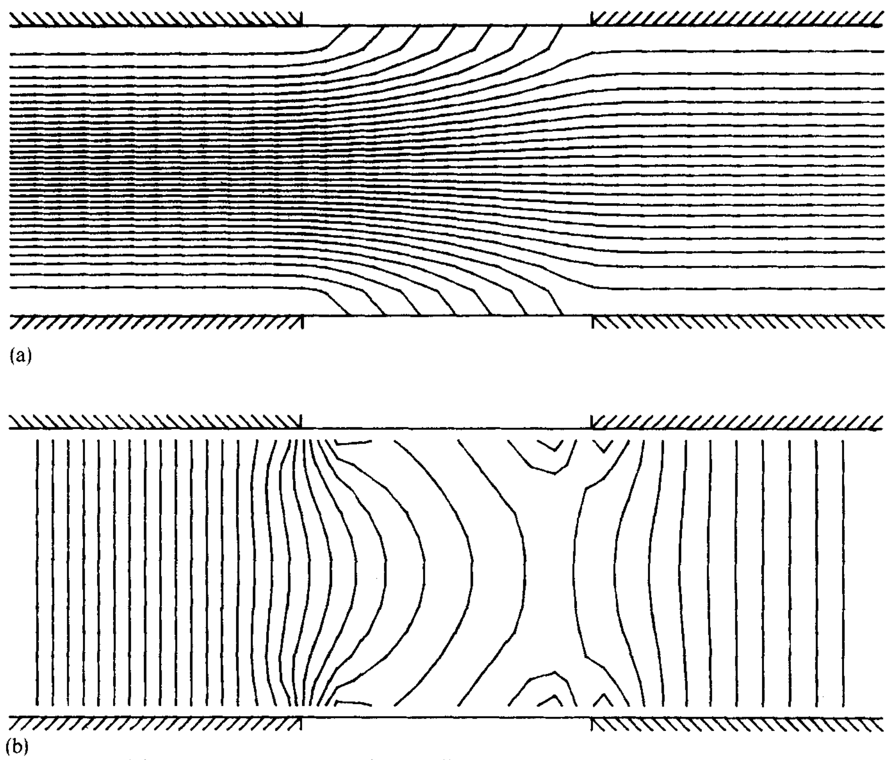

Figure 6. Contour plots of streamlines (a), and pressure (b) for Case 3

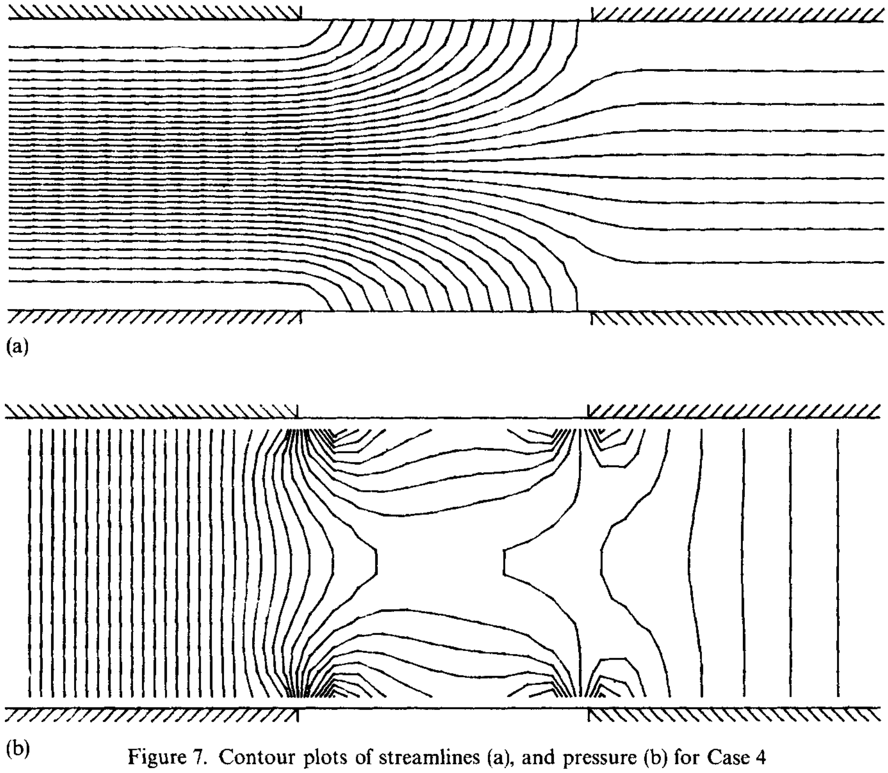

Figure 7. Contour plots of streamlines (a), and pressure (b) for Case 4

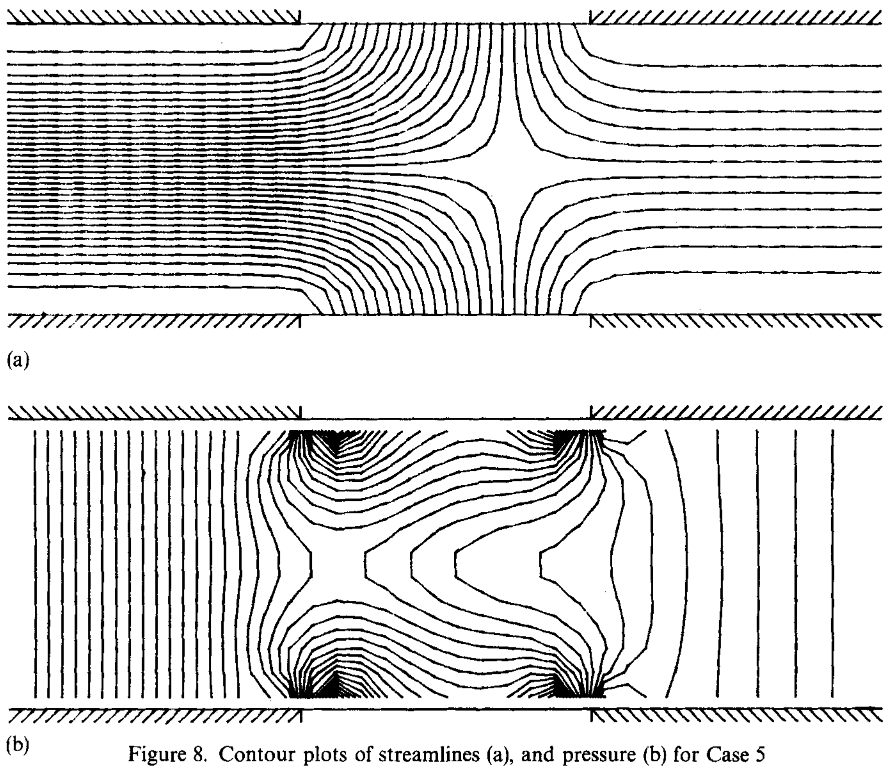

Figure 8. Contour plots of streamlines (a), and pressure (b) for Case 5

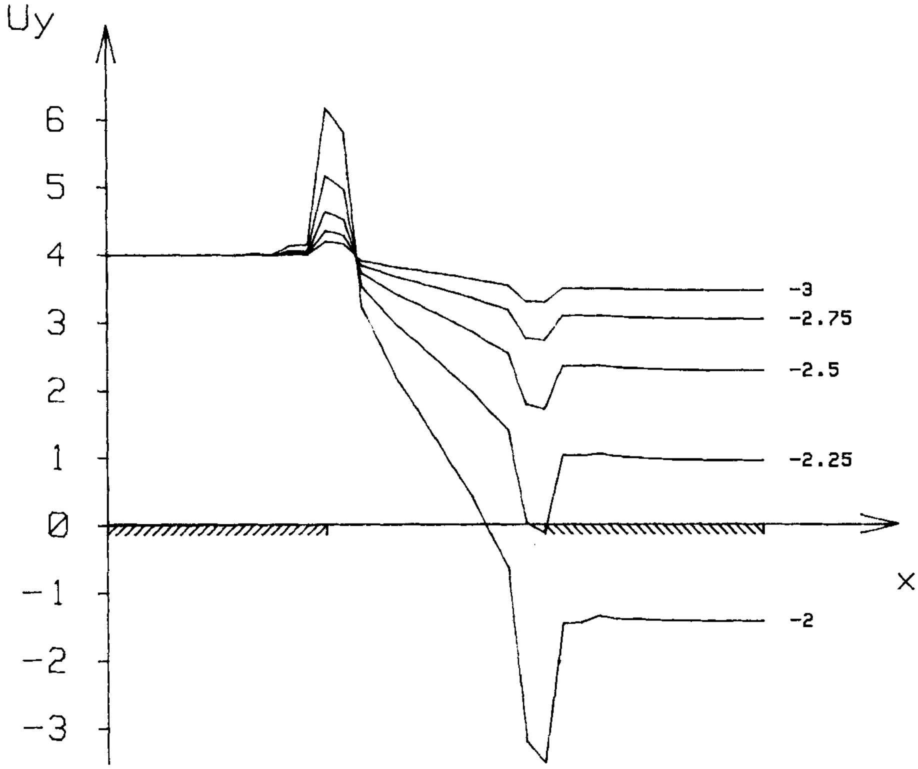

Figure 9. Tangential stress distribution along the wall for different values of log k

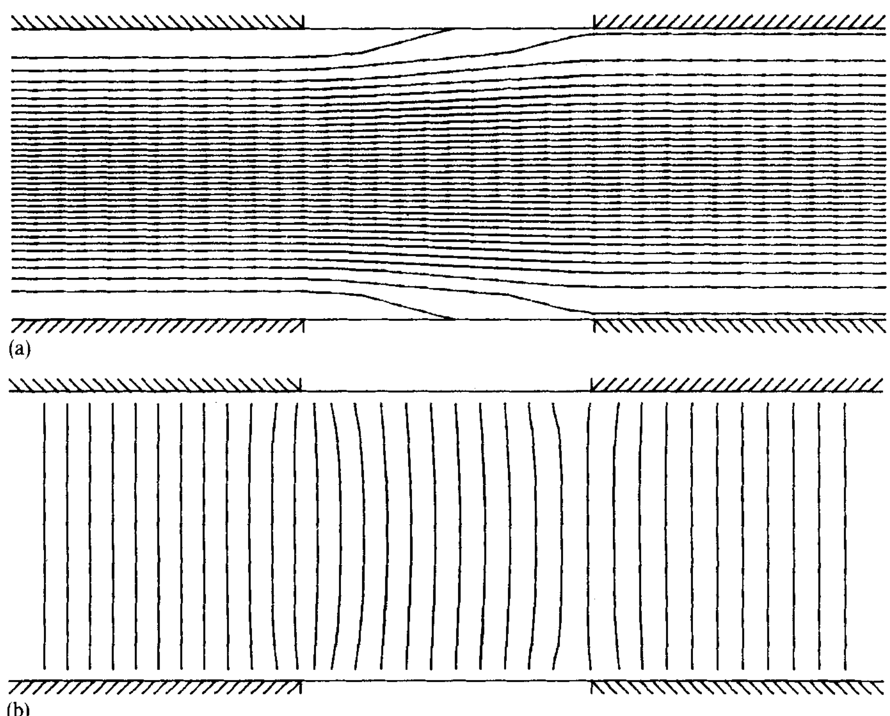

Figure 10. Contour plots of streamlines (a), and pressure (b) for Case 6

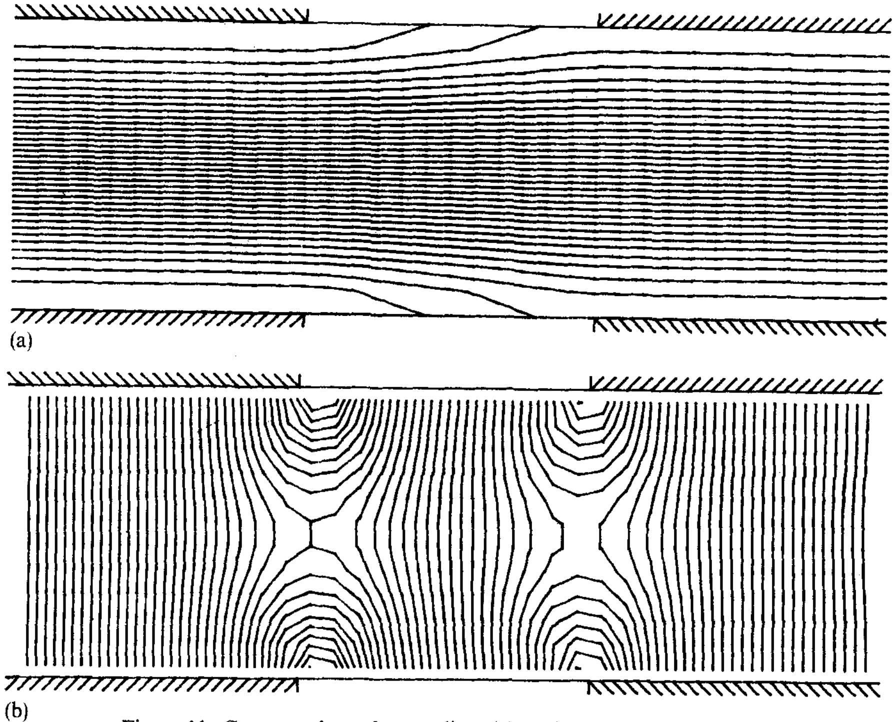

Figure 11. Contour plots of streamlines (a), and pressure (b) for Case 7

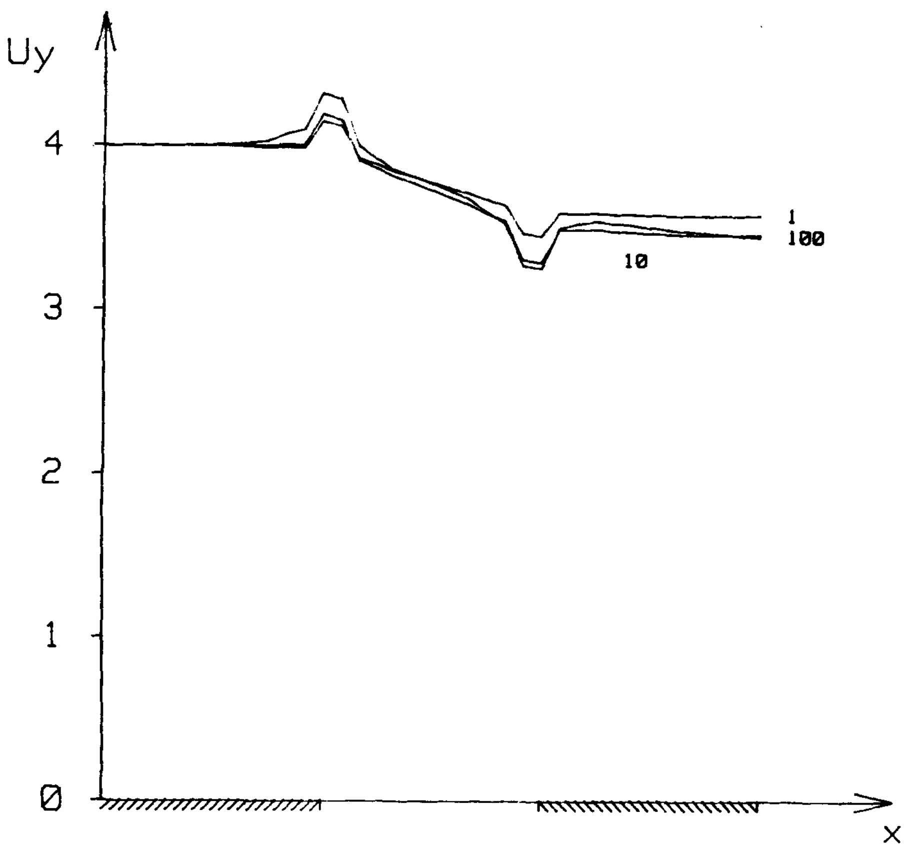

Figure 12. Tangential stress distribution along the wall for different values of *Re*

Regularization

The regularization method has been tested for different values of σ (which is effectively the reciprocal of the Lagrange multiplier) for the conditions corresponding to Case 1. In Figure 13 the CPU time (min) is plotted vs σ. Such a CPU time accounts also for the computational time that is required to determine the optimized SOR factor. Figure 13 shows that the CPU time reaches a minimum for a value of σ around 2. However such a value is only an indication of the efficiency of the method, since in all cases the stiffness of the pressure matrix was not critical (the advantages of the regularization are more evident for meshes finer than the ones employed in the present work).

CONCLUSIONS

In the present work a finite difference method for the solution of the two-dimensional incompressible laminar Navier–Stokes equations, in the presence of permeable walls, has been developed and tested.

First the problem of the (finite difference) pressure treatment in standard and non-standard boundary value NS problems has been critically discussed. Incompatibility effects have been shown to arise owing to an incorrect prescription of the boundary conditions, when one closes the problem of the pressure determination by solving an analytical equation for p. Instead, an algebraic equation (obtained by imposing the conservation of mass in every computational volume) has been shown to be preferable, since a dubious pressure b.c. can be avoided. The method is computationally very efficient. Indeed, the non-classical staggering of pressure and velocity here

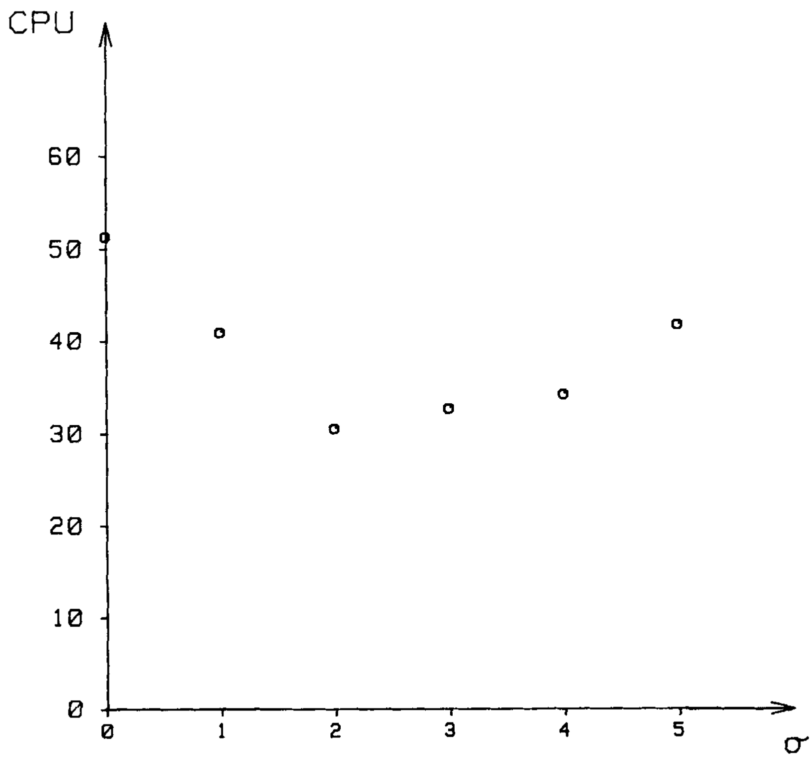

Figure 13. CPU time (in min) vs σ

employed yields a decoupling of the pressure unknowns in even and odd ones, with an obvious reduction in the computational effort.

The algorithm has been tested in simple planar geometries, where inflow/outflow, solid and filtrating wall b.c.s are present. The adequacy of the computed results indicate that the method yields a good description of the steady state. The transient is not properly represented, mainly due to the explicit treatment of the pressure/velocity correlation along the membrane. To eliminate such an artificial delay, an implicit treatment of the filtrating wall b.c. needs to be investigated.

ACKNOWLEDGEMENTS

Part of this work was delivered as an oral presentation at the Third International Conference on Laminar and Turbulent Flows, Seattle, Washington, August 1983, and it was supported by the Ministero della Pubblica Istruzione of Italy.

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